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# Stability of natural convection in a narrow rotating annulus

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The stability of the conduction regime of natural convection of a fluid contained in a narrow rotating annulus with a heated inner wall has been investigated according to the linear theory. The results include computations for Prandtl numbers $P = 0$, 0·72 and 6·7 over a large range of the rotational parameters. For low rotation rates the instability sets in as multicellular convection with the cell axes horizontal in the absence of rotation but tilting monotonically towards the vertical as the rotation rate is increased. Two other types of instability were found at high rotation rates. For large Froude numbers the unstable thermal stratification leads to a Bénard type of convection with vertically oriented rolls. For large Taylor numbers, through a mainly hydrodynamic mechanism, a nearly vertically oriented cellular structure develops in the flow which is destabilized as a result of rotation.

## 1. Introduction

The present work, dealing with the effect of rotation on the stability of natural convection in a fluid contained in an annular region, is an extension of previous studies of natural convection in vertical and inclined rectangular enclosures. In fact, when the annulus is made narrow by taking it to be infinitely tall and of small enough gap width in relation to its mean radius so that the effects of curvature can be neglected, then the convection in vertical and inclined rectangular enclosures become special cases of the analysis reported here.

Besides the question to be answered in a technical application of whether increased heat transfer is obtained by virtue of the basic flow becoming unstable when the annulus is subjected to rotation, the problem being considered may also have some meteorological interest as one limiting case of the flow in a rotating annulus of finite height. The latter problem has received much attention as a model of the earth's atmosphere. The stability aspects of this meteorological problem, however, are beyond the scope of this investigation and are not explored here. A summary of the relevant studies may be found in the monograph by Greenspan (1968).

By taking the annulus as a narrow one we limit our discussion to that regime of the flow in which the transfer of heat takes place entirely by conduction. In this case, in the absence of rotation, the temperature distribution is known to be linear and the velocity profile a cubic function of the co-ordinate perpendicular to the side walls with the fluid flowing up near the hot wall and down next to the cold one. The first two studies in which this, the simplest solution of the equations of motion in this physical situation, has been mentioned are those of Batchelor (1954), who was interested in the heat transfer in tall but finite rectangular enclosures, and Gershuni (1953), who considered the stability of this flow. Batchelor also gave a criterion for how large the aspect ratio, defined here as the height-to-width ratio of the enclosure, should be in order to have the flow in the conduction regime. This has been later modified by Gill & Davey (1969), who estimated that the conduction regime will prevail whenever the aspect ratio $h > \frac{1}{300}R$, where $R$ is the Rayleigh number. Below this value the flow is in the convection regime, which is characterized by an inviscid core with boundary layers on the vertical side walls.

Introduction of rotation with the flow in the convection regime brings in new phenomena. In the interior a circumferential velocity, known in meteorology by the name 'thermal wind', develops which varies linearly with the vertical co-ordinate, as does the temperature. It is the stability of this base flow and the ensuing motion which has been of interest to meteorologists because the wave pattern which develops as a result of the instability resembles the atmospheric jet stream. Prompted by the experimental observations of Fultz (1951, 1953) and Hide (1953, 1958), it has been studied theoretically by Davies (1956, 1959), Kuo (1954, 1956, 1957), Brindley (1960) and Lorenz (1962). While in these studies only the flow in the inviscid core has been taken into account, Barcilon (1964) and O'Neil (1969) have investigated the effect of the Ekman layers on the stability. A yet more realistic model in so far as the simulation of laboratory flows is concerned would also include the vertical boundary layers on the side walls.

In the limit of a very narrow annulus the flow becomes dominated by viscosity. In the absence of an imposed vertical temperature gradient the vertical stratification of temperature disappears as the conduction regime is approached. The circumferential motions thus also vanish and only vertical flow, which is identical to the motion in the absence of rotation, remains, except of course near the top and the bottom parts of the annular region.

The studies of the stability of this flow pattern in the absence of rotation were initiated, as has been mentioned, by Gershuni (1953) and his co-workers Birikh *et al.* (1965, 1973), Birikh (1967) and Rudakov (1967*a*, *b*). Further studies were undertaken by Vest & Arpaci (1969), Gill & Davey (1969), Gill & Kirkham (1970) and by Korpela, Gözüm & Baxi (1973). Some of the asymptotic properties of the solutions have been investigated by Gotoh & Satoh (1966) and Gotoh & Ikeda (1971*a*, *b*, 1972). As a result of these investigations, the criteria for the onset of instability and the ensuing flow patterns are now well determined.

While the correspondence between a rotating annulus and a vertical enclosure is obtained by neglecting entirely the parameters describing the rotation, for the flow in the inclined case only the effect of the Coriolis force is neglected. The gravitational and the centrifugal forces in the rotating flow correspond to the

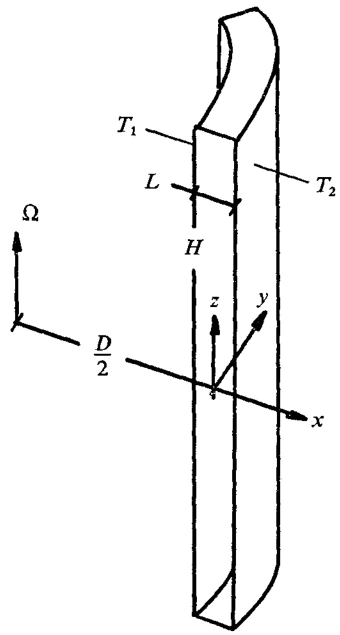


FIGURE 1. A section through a narrow rotating annulus.

two components of the gravitational force in the inclined case. Gershuni (1955) was also the first to attempt a stability analysis of this problem. Later, in the paper by Gershuni & Zhukhovitskii (1969) the analysis was refined and the instability was shown to lead to a horizontal cell structure for small angles of inclination and longitudinal cells for angles closer to the horizontal. The physical mechanism for the instability was discussed later by Hart (1971) in terms of the energy flow in the system. The effect of Prandtl number on the stability was given recently by Korpela (1974).

Another interesting situation arises if the gravity force is neglected while the annulus is allowed to rotate. This case has been discussed by Chandrasekhar (1954) and will be shown in the present Cartesian case to lead to the Rayleigh–Bénard problem. The unstable stratification of temperature in the radial direction is also present in the more general case considered here. Its influence as well as the influence of the Coriolis forces on the stability of the vertical base flow are the main objectives of this study.

## 2. Formulation and solution

An annulus as shown in figure 1 with its dimensions such that $H/L \gg 1$ and $D/L \gg 1$ is considered. With the curvature neglected a Cartesian co-ordinate system is used. This co-ordinate system is fixed at the midplane of the annular gap in such a way that the $z$ co-ordinate is aligned with the axis of the annulus, which is also the axis of rotation, and the co-ordinate system rotates with the angular velocity $\mathbf{\Omega}$. The $+z$ direction is the same as that of the rotation vector, this being the direction opposite to gravity $\mathbf{g}$. The inner and outer walls, at $x = -\frac{1}{2}L$ and $x = +\frac{1}{2}L$ respectively, are taken as isothermal, the outer wall having a temperature $T_2$ which is higher than the inner-wall temperature $T_1$. The temperature level is assumed low enough so that the fluid motion is not influenced by thermal radiation. Alternatively, taking the fluid as a transparent one even if the temperature level were high uncouples the fluid motion from the radiation field. The temperature difference $\Delta T = T_2 - T_1$ is also assumed small to allow the Boussinesq approximation to be used. The kinematic viscosity $\nu$ and the thermal diffusivity $\kappa$ can accordingly be taken as constants and the viscous dissipation and work of compression can be neglected.

Noting that in the case of an infinitely tall annulus the effect of rotation on the base flow is limited to an altered pressure distribution, the base flow velocity and temperature distributions are the same as those in the non-rotating case. These were given by Batchelor (1954) as

$$\frac{\overline{w}}{U} = \frac{1}{6}\left[\frac{1}{4}\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^3\right], \quad \frac{\overline{\theta}}{\Delta T} = \frac{x}{L}, \tag{1}$$

where $\overline{w}$ is the $z$ component of the velocity, $U = g\gamma\Delta T L^2/\nu$ and $\overline{\theta} = T - T_m$, $T_m$ being the average temperature between the walls. Choosing $U$, $\Delta T$ and $L$ as the characteristic velocity, temperature and length respectively, and introducing $L^2/\nu$ as the scale for time and $\rho U^2$ for pressure, where $\rho$ is the density, the dimensionless forms of the equations governing the fluid motions become

$$\nabla . \mathbf{V} = 0, \tag{2}$$

$$\partial \mathbf{V}/\partial t + G(\mathbf{V} . \nabla)\mathbf{V} + T(\hat{\mathbf{k}} \times \mathbf{V}) = -G\nabla p - (F\hat{\mathbf{\imath}} - \hat{\mathbf{k}})\theta + \nabla^2 \mathbf{V}, \tag{3}$$

$$\partial\theta/\partial t + G(\mathbf{V} . \nabla)\theta = P^{-1}\nabla^2\theta. \tag{4}$$

In these equations $\mathbf{V}$ is the non-dimensional velocity and $p$ the non-dimensional pressure measured above the hydrostatic value. The parameters appearing in the equations are defined as $G = UL/\nu$, the Grashof number, $T = 2\Omega L^2/\nu$, the Taylor number, $P = \nu/\kappa$, the Prandtl number, and $F = \Omega^2 D/2g$, the Froude number. As usual, the Taylor number measures the importance of the Coriolis forces and the Froude number the effect of the centrifugal body force, which is taken as constant across the gap, consistent with the approximation $L/D \ll 1$.

The stability of the base flow to disturbances of infinitesimal size is obtained through the study of the equations governing the evolution of these disturbances. These are the linearized forms of (2)–(4),

$$\nabla . \mathbf{V}' = 0, \tag{5}$$

$$\frac{\partial \mathbf{V}'}{\partial t} + G\overline{w}(\hat{\mathbf{k}} . \nabla)\mathbf{V}' + G\frac{d\overline{w}}{dx}(\hat{\mathbf{\imath}} . \mathbf{V}')\hat{\mathbf{k}} + T(\hat{\mathbf{k}} \times \mathbf{V}') = -G\nabla p' - (F\hat{\mathbf{\imath}} - \hat{\mathbf{k}})\theta' + \nabla^2\mathbf{V}', \tag{6}$$

$$\frac{\partial \theta'}{\partial t} + G\overline{w}(\hat{\mathbf{k}} . \nabla)\theta' + G\frac{d\overline{\theta}}{dx}(\hat{\mathbf{\imath}} . \mathbf{V}') = \frac{1}{P}\nabla^2\theta', \tag{7}$$

subject to the boundary conditions

$$\mathbf{V}' = \theta' = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}, \tag{8}$$

where a bar denotes a base flow quantity and a prime a perturbation quantity.

Assuming next that the disturbances have spatial dependence of the form

$$q'(x,y,z,t) = q(x,t)\exp\left[ik(y\sin\phi + z\cos\phi)\right] \tag{9}$$

and eliminating the pressure allows (5)–(8) to be written as

$$\begin{aligned}\partial[(D^2-k^2)\,u]/\partial t + ik\cos\phi\{G[\overline{w}(D^2-k^2) - D^2\overline{w}]\,u\}\\ + ik\cos\phi(T\zeta) = Fk^2\theta - ik\cos\phi(D\theta) + (D^2-k^2)^2\,u,\end{aligned} \tag{10}$$

$$\begin{aligned}\partial\zeta/\partial t + ik\cos\phi(G\overline{w}\zeta) + ik\sin\phi(GD\overline{w}u) - ik\cos\phi(Tu)\\ = ik\theta\sin\phi + (D^2-k^2)\,\zeta,\end{aligned} \tag{11}$$

$$\partial\theta/\partial t + ik\cos\phi(G\overline{w}\theta) + GD\overline{\theta}u = P^{-1}(D^2-k^2)\,\theta, \tag{12}$$

$$u = Du = \zeta = \theta = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}, \tag{13}$$

where $D = \partial/\partial x$, and $u = \hat{\mathbf{\imath}}\,.\,\mathbf{V}$ and $\zeta = \hat{\mathbf{\imath}}\,.\,(\nabla\times\mathbf{V})$ are the $x$ components of the velocity and vorticity respectively.

This form of the stability equations is the most convenient for the study of three-dimensional disturbances. Although it is useful at this point to look for simplifications of the equations in the form of the limiting two-dimensional problems, it is clear that, in the absence of a transformation reducing the problem to an equivalent two-dimensional one, the complete solution must include the treatment of three-dimensional disturbances. In addition, on physical grounds there is no reason to expect the problem to be a two-dimensional one owing mainly to the presence of the Coriolis term. For the two limiting two-dimensional cases no substantial simplifications occur when the disturbances are axisymmetric (independent of the $y$ co-ordinate). On the other hand, when the disturbances are independent of the vertical direction, the equations simplify sufficiently to warrant a separate mention.

Letting $\phi = \frac{1}{2}\pi$ in (10)–(13) reduces them to

$$\partial[(D^2-k^2)\,u]/\partial t = k^2F\theta + (D^2-k^2)^2\,u, \tag{14}$$

$$\partial\zeta/\partial t + ikGD\overline{w}u = ik\theta + (D^2-k^2)\,\zeta, \tag{15}$$

$$\partial\theta/\partial t + GD\overline{\theta}u = P^{-1}(D^2-k^2)\,\theta, \tag{16}$$

$$u = Du = \zeta = \theta = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{17}$$

The great simplification in these equations is due to the decoupling of the vorticity equation (15). The remaining equations are then recognized to be the same as those governing the stability of Rayleigh–Bénard convection, the solution of which can be found in Chandrasekhar (1961, p. 43).

Although the general problem (10)–(13) for three-dimensional disturbances is not self-adjoint, it can, following Roberts (1960), be given a variational basis in terms of its adjoint problem. The eigenvalue problem posed by the variational formulation could then be solved. If the same set of trial functions is used for the original and the adjoint problem the variational method yields the Galerkin equations. These equations can of course also be readily obtained from the original differential equations. In either case, the Galerkin equations, which in turn constitute an algebraic eigenvalue problem, are solvable by numerical methods.

Other similar methods could also be used. One of these, for example, has been extensively employed by Chandrasekhar (1961, p. 300). However, a recent study by Gözüm & Arpaci (1974) of the stability of viscoelastic fluids in a vertical slot implies that this approach is not likely to be advantageous for the present problem. The reason for this is that this approach suggests here the use of an expansion for temperature and the evaluation of velocity in terms of this temperature; and since for $P \to 0$ the stability in this case is known to be entirely hydrodynamic in nature and thus independent of temperature perturbations, difficulties in convergence at small $P$ can be expected.

In the Galerkin method which was adopted the orthogonal series for the velocity was constructed from the $C$ and $S$ functions tabulated by Harris & Reid (1958) while trigonometric functions were used for the vorticity and the temperature perturbations. The expansions can be written as

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, C(\lambda_n x) + i b_n(t)\, S(\mu_n x), \tag{18}$$

$$\zeta(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin \kappa_n x + i e_n(t) \cos \rho_n x, \tag{19}$$

$$\theta(x,t) = \sum_{n=1}^{\infty} f_n(t) \cos \rho_n x + i g_n(t) \sin \kappa_n x, \tag{20}$$

where $\kappa_n = 2n\pi$ and $\rho_n = (2n-1)\pi$. More complex expansions have already been used by Rudakov (1967*a*), and even more complicated ones by Dolph & Lewis (1958) with no apparent advantage. However, an article by Orszag (1971) on the Orr–Sommerfeld equation points to possible merits of using Chebyshev polynomials for considerably improved convergence. Further tests on the use of polynomial expansions for stability analyses appear to be needed.

While the coefficients in (19) and (20) are in general complex, for the case when the instability sets in as steady convection, which includes all our calculated results, writing the expansions in this form ensures that the coefficients are real. Substituting next the expressions (18)–(20) into (10)–(12) and orthogonalizing results in the matrix equation

$$\mathbf{A}\, d\mathbf{X}/dt + \mathbf{B}\mathbf{X} = 0, \tag{21}$$

where $\mathbf{X}$ is the vector of coefficients and $\mathbf{A}$ and $\mathbf{B}$ are matrices containing only real elements. The order of the matrices is $6N$ when the expansions (18)–(20) are truncated to $N$ terms.

The general matrix eigenvalue problem (21) was solved by the QZ algorithm of Moler & Stewart (1971). The algorithm is a generalization of the QR procedure described, for example, in Wilkinson (1965) and reduces to it when $\mathbf{A}$ is the identity matrix. As was mentioned above, this computation was made only infrequently since steady convection was always the outcome of the instability, and it was thus sufficient to find the conditions under which the determinant of the matrix $\mathbf{B}$ vanishes to establish the state of stability of the flow. This was accomplished by using the LU decomposition on the matrix $\mathbf{B}$ (see Forsythe & Moler 1967, p. 68) and finding the determinant as the product of the diagonal terms in the matrix $\mathbf{U}$ with an appropriate algebraic sign included.

| Number of waves | $\eta = D_i/D_o$ | | | | |
| --- | --- | --- | --- | --- | --- |
| $m$ | 0·2 | 0·3 | 0·4 | 0·5 | 0·5† |
| 1 | 3697 | — | — | 12 054 | 12 327 |
| 2 | 2032 | 2130 | — | 3 673 | 3 777 |
| 3 | 2479 | 1929 | 1879 | 2 209 | 2 278 |
| 4 | 3832 | 2423 | 1800 | 1 819 | 1 864 |
| 5 | — | — | 2244 | 2 002 | 2 061 |
| 6 | | | | | |

† This column gives the results from a second-order approximation while the rest of the values correspond to a first-order approximation.

TABLE 1. The critical values for the product $RF$ when gravitational forces are negligible

## 3. Discussion and results

*Centrifugal instability*

When the centrifugal forces are sufficiently large, the flow becomes unstable to disturbances which lead to the formation of cells with their axes oriented in a vertical direction. These form as a solution of (14), (16) and (17) with

$$RF = 1707{\cdot}76, \quad k = 3{\cdot}117, \tag{22}$$

marking the onset of instability in accordance with the Rayleigh theory. This result can be compared with the calculations of Chandrasekhar (1954), who considered a more general problem by including the ratio $L/D$ as a parameter. This required him to carry out the solution to the problem in cylindrical co-ordinates. In making a comparison with his results we note that, if the definition of Froude number were based on the mean radius, the parameter $S$ given by him as

$$S = \Omega^2\gamma D_0^4\,\Delta T/[16\kappa\nu \ln (D_o/D_i)]$$

could be written as

$$S = 2RF/[(1+\eta)(1-\eta)^3 \ln \eta^{-1}],$$

where $\eta = D_i/D_o$ and the subscripts on $D$ refer to the inner and outer cylinders. We reproduce his results in table 1 in terms of the product $RF$. They are limited to the first approximation except in the case $\eta = 0{\cdot}5$, for which the second approximation was carried out. For higher values of $\eta$ the convergence of his method deteriorates further, so that no results, probably owing to the lack of large computer facilities at the time of his writing, were given. Although his results tend towards the value given by (22), a good comparison cannot be made because no data are available for values close to $\eta = 1{\cdot}0$.

Following Barcilon (1964) an approximate relation for the number of waves in the limiting solution can be obtained. In this relation, which is

$$m = \tfrac{1}{2}k(1+\eta)/(1-\eta), \tag{23}$$

the integer closest to $m$ is associated with the number of waves. The value of $k$ in this expression is 3·117. A calculation using this relation shows that the correct number of waves is predicted in each case considered by Chandrasekhar, even for $\eta = 0{\cdot}2$. This is not really surprising since the selection of the critical wave-number appears to be governed basically by the geometry rather than the type of instability. (Compare also with Davies 1959.)

### *Instability for $P \to 0$*

When the Prandtl number is set to zero in (12), the temperature perturbations vanish completely. The instability becomes now entirely hydrodynamic in nature. With the vanishing of the temperature perturbations the dependence on the Froude number vanishes also, so that the only parameters which remain are $G$, $T$, $k$ and $\phi$.

The results for $T = 0$ are given in Korpela *et al.* (1973), where it was found that the instability sets in when $G = 7930$, $k = 2{\cdot}69$ and $\phi = 0$. While the cellular motions which arise here are two-dimensional with their axes horizontal, the effect of rotation is expected to lead to a more complicated pattern with a non-zero value for $\phi$. This is a consequence of the Coriolis forces lying in the plane perpendicular to the rotation vector while the two-dimensional motions in the absence of rotation lie in a plane containing the axis about which the annulus is rotated.

With the values for $T = 0$ known, we next consider the case $T \to \infty$. Here we make the assumption, to be justified *a posteriori*, that, as $T \to \infty$, $\phi \to \frac{1}{2}\pi$. Letting $\epsilon = \frac{1}{2}\pi - \phi$, the governing equations (10)–(13) after elimination of $\zeta$ become

$$(D^2 - k^2)(D^2 - k^2 - \partial/\partial t)^2 u = -k^2 \epsilon T (G D\overline{w} - \epsilon T) u, \quad (24)$$

$$u = Du = (D^2 - k^2)^2 u = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \quad (25)$$

From (24) it is seen that the Taylor number and the angle $\epsilon$ occur only as the product $\epsilon T$, so that the number of independent parameters has been reduced by one in going to this limit. To obtain the critical value of the Grashof number (24) subject to the boundary conditions (25) could now be solved and a search made for the minimum value of $G$ as a function of $k$ and $\epsilon T$. We did not carry out this calculation separately but obtained the results from the full equations by giving the Taylor number a large value, which in the actual computation was $T = 20\,000$, and searching the $k$, $\epsilon$ plane for the minimum of $G$. The value obtained in this manner was $G = 2867{\cdot}14$ for $\epsilon = 2{\cdot}0787 \times 10^{-3}$ and $k = 3{\cdot}123$. Using this preliminary result the neutral-stability curves for intermediate Taylor numbers were calculated.

As is well known, the convergence of the Galerkin method deteriorates with increasing $\alpha G$. To establish the proper number of terms in the expansions (18)–(20) some trial runs were made. These were performed for the case $P = 0$, $\phi = 0$. The results are shown in tables 2 and 3. Since in our calculations the critical Grashof number was never greater than about 8000, an eight-term expansion was considered sufficient to assure a correct value for the last digit before the decimal point. This also allowed a reasonably accurate determination of the wavenumber since, as may be seen in tables 2 and 3, a low-order approximation gives a value which is too high.

| $k$ | $N$ = 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 2·2 | 8322 | 10 253 | 10 974 | 10 989 | 10 996 |
| 2·5 | 7792 | 9 865 | 10 656 | 10 676 | 10 686 |
| 2·6 | 7651 | 9 793 | 10 613 | 10 636 | 10 647 |
| 2·7 | 7526 | 9 751 | 10 603 | 10 630 | 10 643 |
| 2·8 | 7415 | 9 740 | 10 629 | 10 661 | 10 676 |
| 3·0 | 7232 | 9 835 | 10 808 | 10 857 | 10 879 |

TABLE 2. The critical Grashof number as a function of the number of terms in the expansion $N$ and the wavenumber $k$ for $T = 100$, $P = 0$ and $\phi = 0$

| $k$ | $N$ = 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| 2·8 | — | — | — | 60 051 |
| 2·9 | — | — | 59 441 | 59 973 |
| 3·0 | — | 58 672 | 59 362 | 59 995 |
| 3·1 | — | 58 480 | 59 380 | 60 137 |
| 3·2 | — | 58 352 | 59 516 | — |
| 3·3 | 50 755 | 58 294 | — | — |
| 3·4 | 50 376 | 58 314 | — | — |
| 3·6 | 49 757 | 58 618 | — | — |
| 3·8 | 49 647 | — | — | — |
| 4·0 | 49 188 | — | — | — |

TABLE 3. The critical Grashof number as a function of the number of terms in the expansion $N$ and the wavenumber $k$ for $T = 1000$, $P = 0$ and $\phi = 0$

The results of the computations are shown in figures 2, 3 and 4. In figure 2 the variation of the critical Grashof number is shown as a function of the magnitude $k$ of the wave vector for three Taylor numbers. The angle $\phi$ was kept constant on these curves at a value close to the critical one. The dependence of the minimum value of the Grashof number on $k$ was found to depend only weakly on the Taylor number: the values fell between the limits 2·69 and 3·12, the lower one corresponding to the case of no rotation and the higher one being approached as the Taylor number became large. Figure 3 shows the neutral-stability curves for the same Taylor numbers as a function of the angle $\phi$. A constant value of $k = 2{\cdot}65$ is associated with each of the curves. The three-dimensional nature of the problem is well exhibited in this figure as the critical point, corresponding to the minimum of each curve, shifts towards increasing values of $\phi$. This gives validity to the approximations made in deriving (24) and (25). In physical terms this shift is associated with the formation of rolls, as a result of instability, which are oriented horizontally in the absence of rotation, tilting to approximately 45° when $T = 50$ and assuming with a further increase in the Taylor number a predominantly vertical orientation.

The set of critical states is summarized in figure 4. The critical Grashof number, starting from a value of 7930 in the absence of rotation, diminishes to 2867, which is given as an asymptote in the figure for very large values of the Taylor number.

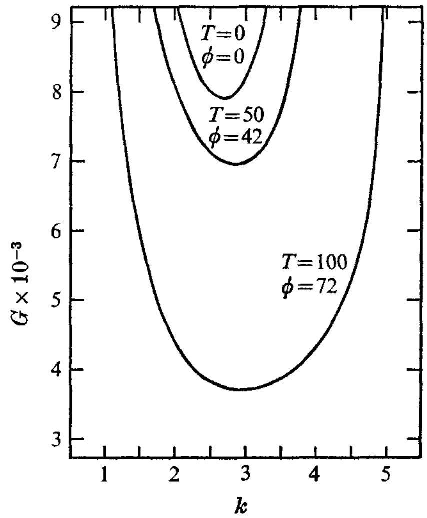


FIGURE 2. Variation of the critical Grashof number for a fluid with $P = 0$ as a function of the magnitude $k$ of the wavenumber vector.

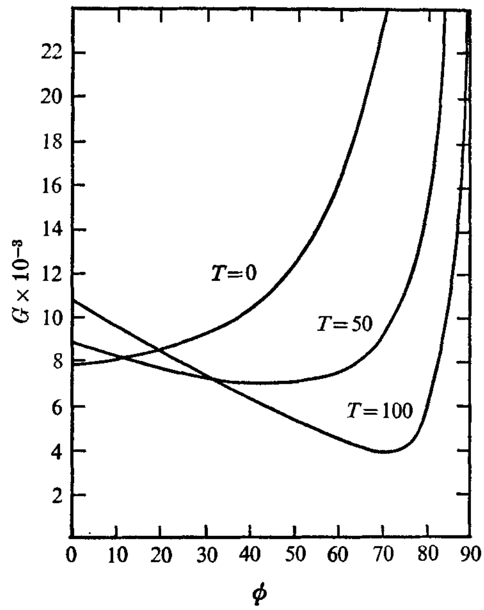


FIGURE 3. Variation of the critical Grashof number for a fluid with $P = 0$ as a function of the phase $\phi$ of the wavenumber vector. $k = 2{\cdot}65$.

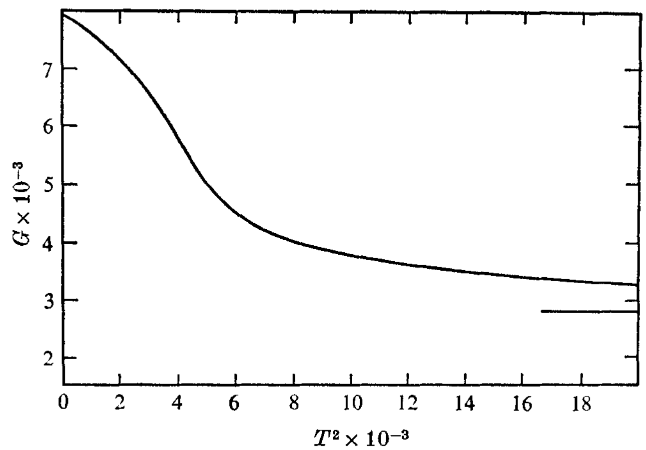


FIGURE 4. The states of neutral stability for $P = 0$.

*Instability for finite P*

For non-zero Prandtl numbers the effect of rotation is manifested in both the Froude and the Taylor number. Since for a given fluid the Prandtl number is fixed, it would seem best to represent the results in co-ordinates with the Taylor and Froude numbers as the abscissae and the Grashof number as the ordinate. The Prandtl number would then remain a parameter for the surfaces of neutral

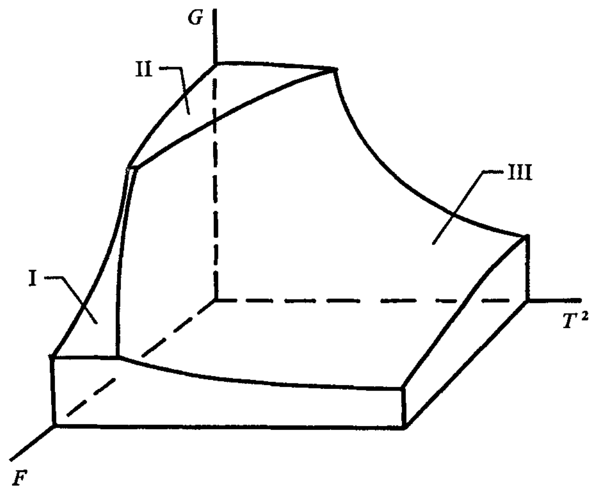


FIGURE 5. A sketch of the neutral-stability surface for $P = 0{\cdot}72$.

stability. Such a stability diagram is given in figure 5. However, if, rather than the Taylor number, the square of the Taylor number is used as a co-ordinate, an increase in the rotation rate, with the other parameters held fixed, corresponds to following a ray from the origin on a constant-$G$ plane in a direction which is compatible with the scales of the horizontal axes. Three types of instability were found when the parameters of the problem were varied. The lower left-hand portion in figure 5 corresponds to centrifugal instability, termed here class I instability. In the range of parameters corresponding to the top portion of the figure the instability is associated primarily with the vorticity distribution of the base flow, and is basically of the type which occurs in the absence of rotation for low Prandtl numbers. This is called class II instability. Finally, on the right side of the figure the Coriolis forces play an important part. This instability is called class III type. For air ($P = 0{\cdot}72$), the surfaces of constant Froude number are shown in figure 6. The point of division of the class I and II instabilities occurs at $F = 0{\cdot}34$ for $T = 0$, the lower values of $F$ corresponding to class II. The transition point between classes II and III is seen to occur at about $T = 33$ for $F = 0$, this point moving to lower Taylor numbers as the Froude number is increased. An idea of how this transition occurs for $F = 0$ is displayed in figure 7, in which the Taylor number is a parameter for the curves. The value of $k$ is held constant at $2{\cdot}20$, which is close to the critical wavenumber, for the entire set. Considering the curves for $T = 35$ and 40, it may be noted that basically two minima occur. The first, corresponding to class II instability, which is quite insensitive to Taylor number variation, appears near $\phi = 25°$. The other, associated with class III instability, depends strongly on the Taylor number and occurs in the neighbourhood of $\phi = 70°$. Between these two instabilities there is a stable region which extends its peak to very large Grashof numbers. On increasing the Taylor number the peak is brought down; indeed for $T = 60$ it has been brought down sufficiently so that one can speak of only one minimum for the curve.

A spectrum of eigenvalues is shown in figure 8 for a typical case near the nose

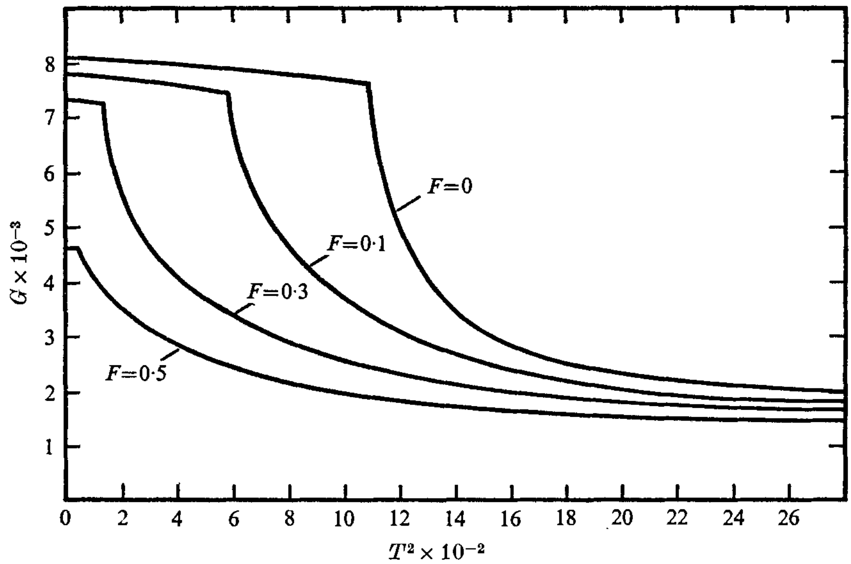


FIGURE 6. The states of neutral stability for $P = 0{\cdot}72$.

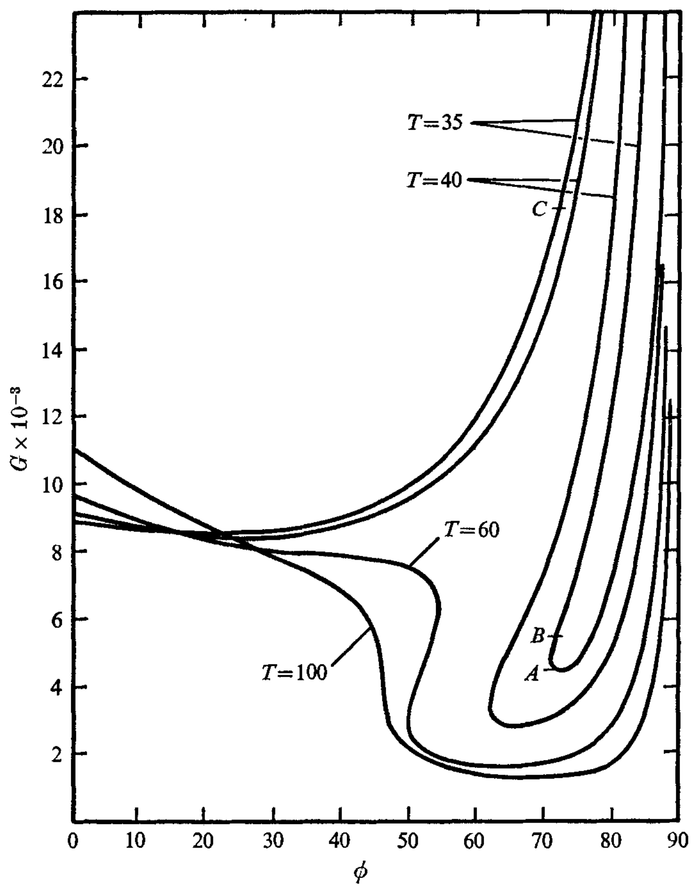


FIGURE 7. The dependence of the critical Grashof number for a fluid with $P = 0{\cdot}72$ on the phase angle $\phi$ in the transition region of class II and class III instabilities. $F = 0, k = 2{\cdot}2$.

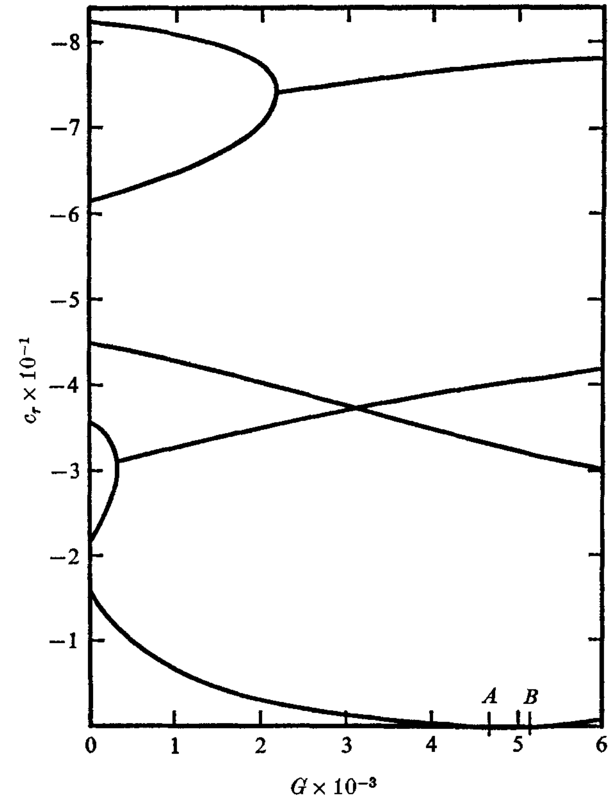


FIGURE 8. The damping rates for a fluid with $P = 0{\cdot}72$ under the conditions $F = 0$, $T = 35$, $\phi = 71{\cdot}5°$ as a function of Grashof number.

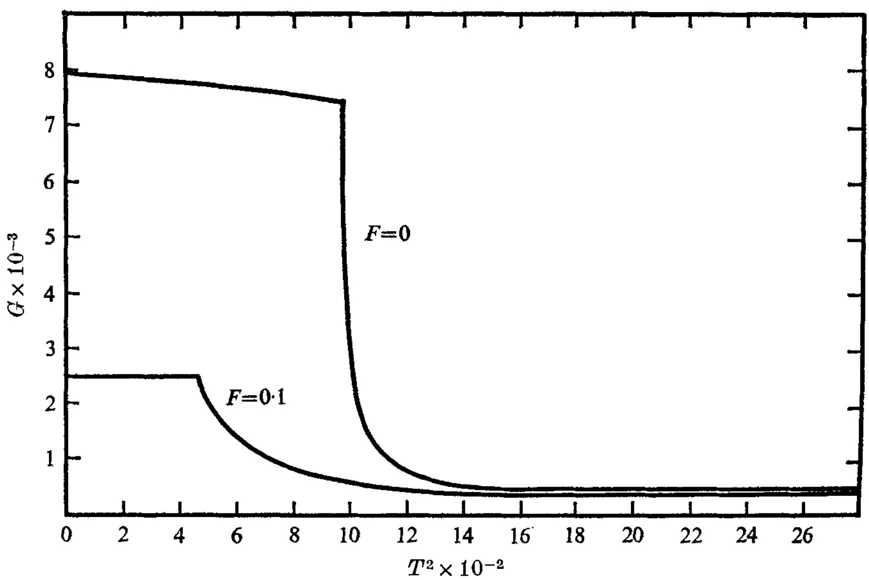


FIGURE 9. The states of neutral stability for $P = 6{\cdot}7$.

of the class III instabilities. The damping rates are plotted as a function of Grashof number for $\phi = 71{\cdot}5°$, $k = 2{\cdot}20$ and $T = 35$. By identifying the eigenvalues by their order at $G = 0$, we note that the lowest one stays real throughout the variation of the Grashof number, becoming positive at point $A$ and, after traversing the unstable region, negative again at point $B$. The second and third eigenvalues combine into a complex conjugate pair at $G = 400$, the real part of which stays in the damped region for all Grashof numbers considered. The fourth eigenvalue is associated with class II instability and becomes critical at point $C$ in figure 7. In figure 7 it can also be seen that on decreasing the Taylor number there is an increase in the minimum value of $G$ for class III instability. This tendency causes finally a shift in the instability to the class II type, the transition point for $F = 0$ being at $T = 33$, as has been mentioned. With this transition there is a discontinuity in the slope of the critical curve, as well as in the value of the angle $\phi$. The critical $k$ also changes abruptly, from 2·20 for class III instability to 2·90 for class II instability. What actually happens in a physical situation depends on the relative amplification rates of the corresponding disturbances and, after the initial growth, on the nonlinear terms of the equations which we neglected in the present linearized study. For water ($P = 6{\cdot}7$) the situation is similar to that for air ($P = 0{\cdot}72$), the only notable difference being that the Froude number effect is more pronounced. The shift between class II and class I instability has already taken place at $F = 0{\cdot}1$. The calculations are summarized in figure 9.

Even in as extensive a parametric study as we have attempted to carry out in this investigation, some aspects of the problem have still been left untreated. In particular, as is the case for a vertical enclosure, for large Prandtl numbers instability in the form of travelling waves is expected to take place. Besides this, no calculations were performed to establish the contributions of the various terms to the energy of the disturbance. An idea how this takes place can be deduced from the results of Hart (1971), who studied the flow in an inclined enclosure. In particular, since the Coriolis forces cannot do work, their effect is only to alter the way in which the transfer of energy from the base flow takes place under the action of the Reynolds stresses; and since the effect of the Taylor number is to destabilize the flow, this transfer is facilitated by rotating the annulus.

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